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SOME REMARKS ON THE CATEGORY SET(L), PART III

Sergey A. Solovyov
University of Latvia, Latvia

Abstract. This paper considers the category SET(L) of L-subsets of sets with a fixed basis L and is a continuation of our previous investigation of this category. Here we study its general properties (e.g., we derive that the category is a topological construct) as well as some of its special objects and morphisms.

1. Introduction

The notion of a fuzzy set introduced in [7] induced many researchers to study different mathematical structures involving fuzzy sets and their generalization L-fuzzy sets [1] or just L-sets for short. In particular, some authors considered the category SET(L) of all L-subsets of sets with a *fixed* basis L. The aim of our work is further contribution to the study of some intrinsic properties of the category SET(L). The article is a continuation of our previous investigation of this category in [5, 6] where we considered some special objects and morphisms as well as some standard constructions in it. Despite of being a continuation the article is self-contained and does not require from the reader to be familiar with the preceding parts.

The paper starts with an introductory section, i.e., Preliminaries, where we recall the definition of the category SET(L) and discuss some results from our previous investigation of this category. The next section is devoted to general properties of the category SET(L). We prove that the category is a topological construct and consider its relations to topoi theory. We continue by considering some special morphisms and objects in the category SET(L). Here we consider some types of monomorphisms (epimorphisms) and derive that they all are equivalent.

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We use standard terminology accepted in Category theory (see, e.g., [3]).

2. Preliminaries

In this section we will discuss some basic properties of the category $\mathrm{SET}(L)$. Let us start by recalling its definition (see [1]).

Suppose L is a complete lattice $(L, \leqslant)$, i.e., a partially ordered set such that for every subset $A \subset L$ the join $\bigvee A$ and the meet $\bigwedge A$ are defined. In particular, $\bigvee L =: 1_L$ and $\bigwedge L =: 0_L$. We assume that $0_L \neq 1_L$, i.e., L has at least two elements. Then the category $\mathrm{SET}(L)$ can be defined as follows.

The objects of $\mathrm{SET}(L)$ are all L-subsets of sets, i.e., mappings $X : \tilde{X} \to L$ where $\tilde{X}$ is an arbitrary set (maybe empty). Henceforth the objects of $\mathrm{SET}(L)$ will be denoted by X, Y or Z and arbitrary sets by $\tilde{X}$, $\tilde{Y}$ or $\tilde{Z}$. By saying that an object $X \in \mathtt{Obj}\,\mathrm{SET}(L)$ is given we will always mean that X is a mapping $X : \tilde{X} \to L$.

Given two objects $X, Y \in \mathtt{Obj}\,\mathrm{SET}(L)$, the set of morphisms from X to Y $\mathtt{Mor}_{\,\mathrm{SET}(L)}(X, Y)$ consists of all mappings $f : \tilde{X} \to \tilde{Y}$ such that $X(x) \leqslant Y \circ f(x)$ for all $x \in \tilde{X}$. Given an object $X \in \mathtt{Obj}\,\mathrm{SET}(L)$, we denote its identity morphism by e_X.

Now we will list some properties of the category $\mathrm{SET}(L)$ which we will need throughout the article and whose proofs can be found in [5, 6]. All of them are related to special morphisms and objects in the category $\mathrm{SET}(L)$; also notice that we use "iff" for "if and only if".

A morphism $f : X \to Y$ is

(1) a monomorphism iff f is injective;
(2) a regular monomorphism iff f is injective and $X = Y \circ f$;
(3) an epimorphism iff f is surjective;
(4) a regular epimorphism iff f is surjective and $Y(y) = \bigvee\{X(x) \mid f(x) = y\}$ for every $y \in \tilde{Y}$;
(5) an isomorphism iff f is bijective and $X(x) = Y \circ f(x)$ for all $x \in \tilde{X}$;

An object X is a final object iff $\tilde{X} = \{x_0\}$ and $X(x_0) = 1_L$.

3. On some general properties of the category $\mathrm{SET}(L)$

In this section we will consider some general properties of the category $\mathrm{SET}(L)$. Let us start with a remark concerning its objects.

Suppose we have some $X \in \mathtt{Obj}\,\mathrm{SET}(L)$. One can consider the map $X : \tilde{X} \to L$ as a structure on the set $\tilde{X}$. Thus, the object X can be viewed upon as a pair $(\tilde{X}, X)$. This gives rise to considering the following notion (see [4]).

Suppose we have a category $\mathcal{C}$. Then $\mathcal{C}$ is called a construct provided that its objects are structured sets, i.e., pairs $(\tilde{X}, \xi)$ where $\tilde{X}$ is a set and ξ is a

$\mathcal{C}$-structure on $\tilde{X}$, and its morphisms $f : (\tilde{X}, \xi) \to (\tilde{Y}, \eta)$ are suitable maps between $\tilde{X}$ and $\tilde{Y}$ whose composition law is the usual composition of maps.

Clearly, one can regard the category SET(L) as a construct. Further, let us consider the following notion (see [4]).

A construct $\mathcal{C}$ is called topological iff it satisfies the following two conditions:

(1) (Existence of initial structures). For any set $\tilde{X}$, any family $((\tilde{X}_i, \xi_i))_{i\in I}$ of $\mathcal{C}$-objects indexed by a class I and any family $(f_i : \tilde{X} \to \tilde{X}_i)_{i\in I}$ of maps indexed by I there exists a unique $\mathcal{C}$-structure ξ on $\tilde{X}$ which is initial with respect to $(\tilde{X}, f_i, (\tilde{X}_i, \xi_i), I)$, i.e., such that for every $\mathcal{C}$-object $(\tilde{Y}, \eta)$ a map $g : (\tilde{Y}, \eta) \to (\tilde{X}, \xi)$ is a $\mathcal{C}$-morphism iff for every $i \in I$ the composite map $f_i \circ g : (\tilde{Y}, \eta) \to (\tilde{X}_i, \xi_i)$ is a $\mathcal{C}$-morphism.

(2) For any set $\tilde{X}$, the class $\{(\tilde{Y}, \eta) \in \mathtt{Obj}\,\mathcal{C} \mid \tilde{Y} = \tilde{X}\}$ of all $\mathcal{C}$-objects with underlying set $\tilde{X}$ is a set.

Notice that if ξ is the initial structure on $\tilde{X}$ with respect to $(\tilde{X}, f_i, (\tilde{X}_i, \xi_i), I)$ then $f_i : (\tilde{X}, \xi) \to (\tilde{X}_i, \xi_i)$ is a $\mathcal{C}$-morphism for each $i \in I$. (Hint. Let $(\tilde{Y}, \eta) = (\tilde{X}, \xi)$ and $g = id_{\tilde{X}}$ in (1)).

THEOREM 3.1. *The category* SET*(L) is a topological construct.*

PROOF. Let us prove that both conditions are fulfilled.

Suppose we have some set $\tilde{X}$, a family $(X_i)_{i\in I}$ of SET(L)-objects and a family $(f_i : \tilde{X} \to \tilde{X}_i)_{i\in I}$ of maps. We have to make an L-set of $\tilde{X}$. Let the map $X : \tilde{X} \to L$ be the following, $X(x) = \bigwedge\limits_{i\in I} X_i \circ f_i(x)$ or just 1_L if $I = \varnothing$ for $x \in \tilde{X}$. Clearly, $X \in \mathtt{Obj}\,\mathrm{SET}(L)$. Further, take any map $f_{i_0} : \tilde{X} \to \tilde{X}_{i_0}$, $i_0 \in I$. Then for every $x_0 \in \tilde{X}$,

$$X(x_0) = \bigwedge_{i\in I} X_i \circ f_i(x_0) \leqslant X_{i_0} \circ f_{i_0}(x_0)$$

and therefore $f_{i_0} \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, X_{i_0})$. Let us prove that the structure on $\tilde{X}$ is initial with respect to $(\tilde{X}, f_i, X_i, I)$.

Suppose we have some $Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ and a map $g : \tilde{Y} \to \tilde{X}$. If $g \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Y, X)$ then obviously $f_i \circ g \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Y, X_i)$ for all $i \in I$ since all f_i are morphisms. Suppose $f_i \circ g \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Y, X_i)$ for all $i \in I$. We have to prove that $g \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Y, X)$. Take any $y_0 \in \tilde{Y}$. Then $Y(y_0) \leqslant X_i \circ f_i \circ g(y_0)$ for all $i \in I$ and therefore

$$Y(y_0) \leqslant \bigwedge_{i\in I} X_i \circ f_i \circ g(y_0) = X \circ g(y_0).$$

Thus, g is indeed a morphism.

Now we will prove the uniqueness of the structure X. Suppose we have another map $X' : \tilde{X} \to L$ which is initial with respect to $(\tilde{X}, f_i, X_i, I)$. Then

$f_i \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X', X_i)$ and therefore $X'(x_0) \leqslant X_i \circ f_i(x_0)$ for every $x_0 \in \tilde{X}$. Thus, $X'(x_0) \leqslant \bigwedge_{i\in I} X_i \circ f_i(x_0) = X(x_0)$. Now let $Y = X$ and let $g = id_{\tilde{X}}$. Since $f_i \circ g = f_i \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, X_i)$ for $i \in I$ then $g \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, X')$ and therefore $X(x_0) \leqslant X' \circ g(x_0) = X'(x_0)$. Thus, $X(x) = X'(x)$ for every $x \in \tilde{X}$ and the initial structure X on the set $\tilde{X}$ is unique.

One can easily verify that for any set $\tilde{X}$ the class $\{Y \in \mathtt{Obj}\,\mathrm{SET}(L) | \tilde{Y} = \tilde{X}\}$ is a set since there is only a set of different maps from $\tilde{X}$ to L. □

Notice that in case of an empty class I we get an indiscrete structure on the set $\tilde{X}$, i.e., $X(x) = 1_L$ for $x \in \tilde{X}$.

Since $\mathrm{SET}(L)$ is a topological construct it has final structures, i.e., for any set $\tilde{X}$, any family $(X_i)_{i\in I}$ of $\mathrm{SET}(L)$-objects indexed by some class I and any family $(f_i : \tilde{X}_i \to \tilde{X})_{i\in I}$ of maps indexed by I there exists a unique $\mathrm{SET}(L)$-structure X on $\tilde{X}$ such that for any $Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ a map $g : \tilde{X} \to \tilde{Y}$ is a $\mathrm{SET}(L)$-morphism iff for every $i \in I$ the composite map $g \circ f_i : \tilde{X}_i \to \tilde{Y}$ is a $\mathrm{SET}(L)$-morphism.

Indeed, suppose we have some set $\tilde{X}$. Let $X(x) = \bigvee_{i\in I} (\bigvee X_i \circ f_i^{-1}(x))$ for $x \in \tilde{X}$ where $\bigvee \varnothing := 0_L$.

Theorem 3.2. *The map X is a final structure on the set $\tilde{X}$ with respect to $(X_i, f_i, \tilde{X}, I)$.*

Proof. Clearly, $X \in \mathtt{Obj}\,\mathrm{SET}(L)$. Further, take any map $f_{i_0} : \tilde{X_{i_0}} \to \tilde{X}$, $i_0 \in I$. Then for every $x_0 \in \tilde{X_{i_0}}$,

$$X_{i_0}(x_0) \leqslant \bigvee X_{i_0} \circ f_{i_0}^{-1} \circ f_{i_0}(x_0) \leqslant \bigvee_{i\in I} (\bigvee X_i \circ f_i^{-1} \circ f_{i_0}(x)) = X \circ f_{i_0}(x_0)$$

and thus, $f_{i_0} \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X_{i_0}, X)$. Now suppose we have some $Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ and a map $g : \tilde{X} \to \tilde{Y}$. If $g \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, Y)$ then obviously $g \circ f_i \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X_i, Y)$ for all $i \in I$ since all f_i are morphisms. Suppose $g \circ f_i \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X_i, Y)$ for $i \in I$. We have to prove that $g \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, Y)$. Take any $x_0 \in \tilde{X}$. If $f_i^{-1}(x_0) = \varnothing$ for all $i \in I$ then $X(x_0) = 0_L \leqslant Y \circ g(x_0)$. Suppose we have some $i_0 \in I$ such that $f_{i_0}^{-1}(x_0) \neq \varnothing$. Then for every $y_0 \in f_{i_0}^{-1}(x_0)$, $X_{i_0}(y_0) \leqslant Y \circ g \circ f_{i_0}(y_0) = Y \circ g(x_0)$. Thus, $\bigvee X_{i_0} \circ f_{i_0}^{-1}(x_0) \leqslant Y \circ g(x_0)$ and therefore

$$X(x_0) = \bigvee_{i\in I} (\bigvee X_i \circ f_i^{-1}(x_0)) \leqslant Y \circ g(x_0).$$

Thus, g is indeed a morphism.

Suppose we have another map $X' : \tilde{X} \to L$ which is final with respect to $(X_i, f_i, \tilde{X}, I)$. Then $f_i \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X_i, X')$ (take $Y = X'$ and let $g = id_{\tilde{X}}$). Further, take some $x_0 \in \tilde{X}$. For every $i \in I$ such that $f_i^{-1}(x_0) \neq \varnothing$ it follows that $X'(x_0) = X' \circ f_i(y_0) \geqslant X_i(y_0)$ for all $y_0 \in f_i^{-1}(x_0)$ and therefore $X'(x_0) \geqslant \bigvee X_i \circ f_i^{-1}(x_0)$. Thus, $X'(x_0) \geqslant \bigvee\limits_{i \in I} (\bigvee X_i \circ f_i^{-1}(x_0)) = X(x_0)$. Therefore, $X'(x) \geqslant X(x)$ for all $x \in \tilde{X}$ since $X(x_0) = 0_L \leqslant X'(x_0)$ for all $x_0 \in \tilde{X}$ such that $f_i^{-1}(x_0) = \varnothing$ for $i \in I$. Now let $Y = X$ and let $g = id_{\tilde{X}}$. Since $g \circ f_i = f_i \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X_i, X)$ for all $i \in I$ then g must be a SET(L)-morphism. Take any $x_0 \in \tilde{X}$. Then $X(x_0) = X \circ g(x_0) \geqslant X'(x_0)$. Thus, $X(x) = X'(x)$ for every $x \in \tilde{X}$ and therefore the final structure X on the set $\tilde{X}$ is unique. □

Notice that in case of an empty class I we get a discrete structure on the set $\tilde{X}$, i.e., $X(x) = 0_L$ for $x \in \tilde{X}$.

Now we will consider a full subcategory $\mathrm{SET}(L)^n$ of SET(L) consisting of normed L-sets, i.e., $\bigvee\limits_{x \in \tilde{X}} X(x) = 1_L$ for every $X \in \mathtt{Obj}\,\mathrm{SET}(L)^n$.

Proposition 3.3. *The category* $\mathrm{SET}(L)^n$ *is not a topological construct.*

Proof. Suppose $\tilde{X} = \{x_0\}$, $\tilde{X}_1 = \{x_1, x_2\}$ and $X(x_1) = 1_L$, $X(x_2) = 0_L$. Clearly, $X_1 \in \mathtt{Obj}\,\mathrm{SET}(L)^n$. Let $f_1(x_0) = x_2$. Then the only map $X : \tilde{X} \to L$ for which f_1 is a morphism is $X(x_0) = 0_L$ and therefore $X \notin \mathtt{Obj}\,\mathrm{SET}(L)^n$. Thus, $\tilde{X}$ has no initial structure with respect to $(\tilde{X}, f_1, X_1, \{1\})$ and therefore the category $\mathrm{SET}(L)^n$ is not a topological construct. □

Now, suppose we have a family $(X_i)_{i \in I}$ of SET(L)-objects indexed by a set I. Let $\tilde{Y} = \prod\limits_{i \in I} \tilde{X}_i = \{(x_i)_{i \in I} \mid x_i \in \tilde{X}_i\}$ and let $Y((x_i)_{i \in I}) = \bigwedge\limits_{i \in I} X_i(x_i)$. For every $i_0 \in I$ let $\pi_{i_0} : \tilde{Y} \to \tilde{X}_{i_0}$ be the projective map, i.e., $\pi_{i_0}((x_i)_{i \in I}) = x_{i_0}$. One can easily verify that $(Y, (\pi_i)_{i \in I})$ is the product of the family $(X_i)_{i \in I}$ (for more details see [5] where we considered the case when $I = \{1, 2\}$). Thus, the following theorem holds.

Theorem 3.4. *The category* SET*(L) has products, i.e., for every set I, each family of* SET*(L)-objects indexed by I has a* SET*(L)-product.*

Further, let $\tilde{Y} = \bigcup\limits_{i \in I} \tilde{X}_i \times \{i\}$ and let $Y((x_i, i)) = X_i(x_i)$ for all $(x_i, i) \in \tilde{Y}$. For every $i_0 \in I$ let $q_{i_0} : \tilde{X}_{i_0} \to \tilde{Y}$ be the inclusion map, i.e., $q_{i_0}(x) = (x, i_0)$. One can easily verify that $((q_i)_{i \in I}, Y)$ is the coproduct of the family $(X_i)_{i \in I}$ (for more details see [5] where we considered the case when $I = \{1, 2\}$). Thus, the following theorem holds.

Theorem 3.5. *The category* SET*(L) has coproducts, i.e., for every set I, each family of* SET*(L)-objects indexed by I has a* SET*(L)-coproduct.*

In [6] we have proved that the category $\mathrm{SET}(L)$ has both equalizers and coequalizers. Since every category which has products and equalizers is complete and dually the category which has coproducts and coequalizers is cocomplete, the following theorem holds.

THEOREM 3.6. *The category* $\mathrm{SET}(L)$ *is both complete and cocomplete.*

Now let us consider the following notion (see [2]). Suppose we have a category $\mathcal{C}$. Then $\mathcal{C}$ is said to allow exponentiation provided that every two $\mathcal{C}$-objects have a product and for every $X, Y \in \mathtt{Obj}\,\mathcal{C}$ there exist an object $Y^X \in \mathtt{Obj}\,\mathcal{C}$ and a morphism $ev : Y^X \times X \to Y$ such that for every $Z \in \mathtt{Obj}\,\mathcal{C}$ and every morphism $g : Z \times X \to Y$ there exists a unique morphism $\hat{g} : Z \to Y^X$ such that the triangle

$$\begin{tikzcd}
Y^X \times X \arrow[dr, "ev"] & \\
Z \times X \arrow[u, dashed, "\hat{g} \times e_X"] \arrow[r, "g"'] & Y
\end{tikzcd}$$

commutes. (Notice that $\hat{g} \times e_X$ denotes the product of the morphisms $\hat{g}$ and e_X.)

Suppose a lattice L' is infinitely distributive, i.e., for every $b \in L'$ and every subset $A \subset L'$, $b \wedge (\bigvee A) = \bigvee\limits_{a \in A} (b \wedge a)$. Then the following theorem holds.

THEOREM 3.7. *The category* $\mathrm{SET}(L')$ *allows exponentiation.*

PROOF. Suppose we have some $X, Y \in \mathtt{Obj}\,\mathrm{SET}(L')$. Let $\widetilde{Y^X}$ be the set of all maps from $\tilde{X}$ to $\tilde{Y}$, i.e., $\widetilde{Y^X} = \{f \mid f : \tilde{X} \to \tilde{Y}\}$ and let $Y^X : \widetilde{Y^X} \to L'$ be the following, $Y^X(f) = \bigvee\{a \in L' \mid X(x) \wedge a \leqslant Y \circ f(x),\, x \in \tilde{X}\}$. Take any $x_0 \in \tilde{X}$. Then $X(x_0) \wedge Y^X(f) = X(x_0) \wedge (\bigvee A) = \bigvee\limits_{a \in A} (X(x_0) \wedge a)$. Since $X(x_0) \wedge a \leqslant Y \circ f(x_0)$ for all $a \in A$ then $\bigvee\limits_{a \in A} (X(x_0) \wedge a) \leqslant Y \circ f(x_0)$ and therefore $Y^X(f) \wedge X(x_0) \leqslant Y \circ f(x_0)$. Thus, $Y^X(f) \wedge X(x) \leqslant Y \circ f(x)$ for $x \in \tilde{X}$. Further, let $ev : \widetilde{Y^X \times X} \to \tilde{Y}$ be the following, $ev(f, x) = f(x)$ for $(f, x) \in \widetilde{Y^X \times X}$. Let us verify that $ev \in \mathtt{Mor}_{\,\mathrm{SET}(L')}(Y^X \times X, Y)$. Indeed, suppose we have some $(f_0, x_0) \in \widetilde{Y^X \times X}$. Then $Y \circ ev(f_0, x_0) = Y \circ f_0(x_0) \geqslant Y^X(f_0) \wedge X(x_0) = Y^X \times X(f_0, x_0)$. Thus, ev is indeed a morphism.

Suppose we have some $Z \in \mathtt{Obj}\,\mathrm{SET}(L')$ and some $g \in \mathtt{Mor}_{\,\mathrm{SET}(L')}(Z \times X, Y)$. Let $\hat{g} : \tilde{Z} \to \widetilde{Y^X}$ be the following, $\hat{g}(z) = g(z, \_)$. Let us verify that $\hat{g} \in \mathtt{Mor}_{\,\mathrm{SET}(L')}(Z, Y^X)$. Suppose we have some $z_0 \in \tilde{Z}$. Then $Y^X \circ \hat{g}(z_0) =$

$Y^X(g(z_0,\, \_))$. Since $g \in \mathtt{Mor}_{\,\mathrm{SET}(L')}(Z \times X,\, Y)$, $Y \circ g(z_0,\, x) \geqslant Z \times X(z_0,\, x) = Z(z_0) \wedge X(x)$ for all $x \in \tilde{X}$. Thus, $Y^X(g(z_0,\, \_)) = \bigvee\{a \in L' \mid X(x) \wedge a \leqslant Y \circ g(z_0,\, x),\, x \in \tilde{X}\} \geqslant Z(z_0)$ and therefore $\hat{g}$ is indeed a morphism.

Now, we have a morphism $\hat{g} \times e_X \in \mathtt{Mor}_{\,\mathrm{SET}(L')}(Z \times X,\, Y^X \times X)$. From our investigation of the category SET(L) in [5] it follows that $\hat{g} \times e_X(\_\,,\, \_) = (\hat{g}(\_)\,,\, e_X(\_))$. Thus $\hat{g} \times e_X(z,\, x) = (\hat{g}(z),\, e_X(x)) = (g(z,\, \_),\, x)$ for $(z,\, x) \in \widetilde{Z \times X}$. One can easily see that the above-mentioned diagram commutes and the morphism $\hat{g}$ is unique. ∎

The next proposition shows one essential property of the category SET(L).

Proposition 3.8. *The category* SET*(L) is not a topos.*

Proof. Suppose the category SET(L) is a topos. Then it has a subobject classifier (see [2]), i.e., a pair $(t,\, \Omega)$ with the following properties:

(1) $t \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(F,\, \Omega)$ where F is a final object in the category SET(L);
(2) (Ω-axiom.) For every two objects $X,\, Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ and every monomorphism $f \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X,\, Y)$ there exists a unique morphism $\chi_f \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Y,\, \Omega)$ such that the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, dashed, "\chi_f"] \\ F \arrow[r, "t"'] & \Omega \end{tikzcd}$$

is a pullback square. (Notice that h is the unique morphism from the set $\mathtt{Mor}_{\,\mathrm{SET}(L)}(X,\, F)$.)

Since F is a final object then $\tilde{F} = \{w_0\}$ and $F(w_0) = 1_L$. Further, suppose $\tilde{X} = \{x_0\}$, $X(x_0) = 0_L$ and $\tilde{Y} = \{y_0\}$, $Y(y_0) = 1_L$. Let $f(x_0) = y_0$. Obviously, $f \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X,\, Y)$ and is injective. Thus, f is a monomorphism. Suppose we have some $\chi_f \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Y,\, \Omega)$ such that the above-mentioned diagram is a pullback square. Let $\tilde{Z} = \{z_0\}$, $Z(z_0) = 1_L$ and let $g_1(z_0) = w_0$, $g_2(z_0) = y_0$. Clearly, $g_1 \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Z,\, F)$, $g_2 \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Z,\, Y)$ and $t \circ g_1(z_0) = t \circ h(x_0) = \chi_f \circ f(x_0) = \chi_f \circ g_2(z_0)$. Since $\mathtt{Mor}_{\,\mathrm{SET}(L)}(Z,\, X) = \varnothing$, the above-mentioned diagram is not a pullback square that contradicts our former assumption. Thus, the category SET(L) has no subobject classifier and therefore is not a topos. ∎

The following theorem shows that though the category SET(L) has no subobject classifier, it has something rather similar to it.

THEOREM 3.9. *There exists a triple* (Ω_1, t, Ω_2), *where* $\Omega_1, \Omega_2 \in \mathtt{Obj}\,\mathrm{SET}(L)$ *and* $t \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(\Omega_1, \Omega_2)$ *such that for every two objects* $X, Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ *and every monomorphism* $f \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, Y)$ *there exists a morphism* $g \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, \Omega_1)$ *with the following property:*

(i) *there exists a unique morphism* $\chi_f \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Y, \Omega_2)$ *such that the square*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, dashed, "\chi_f"] \\ \Omega_1 \arrow[r, "t"'] & \Omega_2 \end{tikzcd}$$

is a pullback square.

PROOF. We will construct the objects Ω_1 and Ω_2 first. Let $\tilde{\Omega}_1 = L$, $\Omega_1(\omega) = \omega$ for all $\omega \in \tilde{\Omega}_1$ and let $\tilde{\Omega}_2 = L \bigcup \{a_*\}$, $a_* \notin L$, $\Omega_2(\omega) = 1_L$ for all $\omega \in \tilde{\Omega}_2$. Further, let $t : \Omega_1 \to \Omega_2$ be the inclusion map, i.e., $t(\omega) = \omega$ for all $\omega \in \tilde{\Omega}_1$. Clearly, $t \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(\Omega_1, \Omega_2)$.

Suppose we have some $X, Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ and a monomorphism $f \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, Y)$. Let $g : \tilde{X} \to \tilde{\Omega}_1$ be the following, $g(x) = X(x)$ for $x \in \tilde{X}$. Since $\Omega_1 \circ g(x) = g(x) = X(x) \geqslant X(x)$ then $g \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, \Omega_1)$. Further, let $\chi_f : \tilde{Y} \to \tilde{\Omega}_2$ be the following,

$$\chi_f(y) = \begin{cases} X \circ f^{-1}(y), & y \in f(\tilde{X}) \\ a_*, & y \notin f(\tilde{X}). \end{cases}$$

Since f is a monomorphism, the map χ_f is defined correctly. Clearly, $\chi_f \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Y, \Omega_2)$. Let us prove that the above-mentioned square is a pullback square. First of all we have to verify that the diagram commutes. Indeed, for every $x_0 \in \tilde{X}$, $t \circ g(x_0) = X(x_0)$ and $\chi_f \circ f(x_0) = X \circ f^{-1} \circ f(x_0) = X(x_0)$. Now suppose we have some $Z \in \mathtt{Obj}\,\mathrm{SET}(L)$ and two morphisms $g_1 \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Z, Y)$ and $g_2 \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Z, \Omega_1)$ such that $\chi_f \circ g_1 = t \circ g_2$. We have to prove that there exists a unique morphism $m \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Z, X)$ such that $g_1 = f \circ m$ and $g_2 = g \circ m$. Let $m : \tilde{Z} \to \tilde{X}$ be the following, $m(z) = f^{-1} \circ g_1(z)$ for $z \in \tilde{Z}$. Since $\chi_f \circ g_1 = t \circ g_2$ then $g_1(\tilde{Z}) \subset f(\tilde{X})$ and therefore m is defined correctly. Further, $f \circ m = f \circ f^{-1} \circ g_1 = g_1$ and $g_2 = t \circ g_2 = \chi_f \circ g_1 = X \circ f^{-1} \circ g_1 = g \circ m$. Now let us verify that $m \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Z, X)$. Indeed, for every $z_0 \in \tilde{Z}$, $X \circ m(z_0) = g \circ m(z_0) = g_2(z_0) = \Omega_1 \circ g_2(z_0) \geqslant Z(z_0)$ since g_2 is a morphism. Lastly, let us verify that the morphism m is unique. Suppose we have another $m' : Z \to X$ such that $f \circ m' = g_1$. Then $m' = f^{-1} \circ g_1 = m$ and therefore the above-mentioned square is indeed a pullback square.

Suppose we have another morphism ${\chi_f}'$. Since ${\chi_f}' \circ f = t \circ g$ then for every $y_0 = f(x_0) \in f(\tilde{X})$ we have $t \circ g(x_0) = X(x_0) = {\chi_f}' \circ f(x_0)$ and therefore ${\chi_f}'(y_0) = X \circ f^{-1}(y_0)$. Thus, $\chi_f|_{f(\tilde{X})} = {\chi_f}'|_{f(\tilde{X})}$. Suppose ${\chi_f}'(y_0) = b_0 \neq a_*$ for some $y_0 \in \tilde{Y} \backslash f(\tilde{X})$. Then let $\tilde{Z} = \{z_0\}$, $Z(z_0) = 0_L$, $g_1(z_0) = y_0$ and $g_2(z_0) = b_0$. Clearly, $g_1 \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Z, Y)$, $g_2 \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Z, \Omega_1)$ and $t \circ g_2(z_0) = b_0 = {\chi_f}'(y_0) = {\chi_f}' \circ g_1(z_0)$. One can easily see that for all maps $m : \tilde{Z} \to \tilde{X}$, $f \circ m(z_0) \neq y_0 = g_1(z_0)$. Thus, the square is not a pullback square that contradicts out former assumption. Therefore, χ_f is the unique morphism with the required property. ∎

The next theorem shows one property of the triple (Ω_1, t, Ω_2).

THEOREM 3.10. *For every two subobjects* (X, f), (Z, h) *of* Y, $(X, f) \approx (Z, h)$ *iff there exists such* $g_f \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, \Omega_1)$ *and* $g_h \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Z, \Omega_1)$ *defined in the previous theorem that* $\chi_f = \chi_h$.

PROOF. We will prove the necessity first and therefore assume that $(X, f) \approx (Z, h)$. Then there exists a unique isomorphism $k : Z \to X$ such that the triangle

$$\begin{tikzcd} Z \arrow[d, dashed, "k"'] \arrow[dr, "h"] & \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. In order to prove that $\chi_f = \chi_h$ we will show that the square

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "g_h"'] & Y \arrow[d, "\chi_f"] \\ \Omega_1 \arrow[r, "t"'] & \Omega_2 \end{tikzcd}$$

is a pullback square. From the uniqueness of χ_h it will immediately follow that $\chi_f = \chi_h$.

First of all we have to verify that the square commutes. Indeed, since $\chi_f \circ f = t \circ g_f$ then $t \circ g_f \circ k = \chi_f \circ f \circ k = \chi_f \circ h$. Further, since k is an isomorphism then $X \circ k(z_0) = Z(z_0)$ for every $z_0 \in \tilde{Z}$ and therefore $g_f \circ k(z_0) = X \circ k(z_0) = Z(z_0) = g_h(z_0)$. Thus, $g_f \circ k = g_h$ and therefore $t \circ g_h = \chi_f \circ h$.

For every $W \in \mathtt{Obj}\,\mathrm{SET}(L)$ and every morphisms $h_1 \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(W, \Omega_1)$ and $h_2 \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(W, Y)$ such that $t \circ h_1 = \chi_f \circ h_2$ there exists a unique morphism $m \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(W,\,X)$ such that the diagram

$$
\begin{tikzcd}
W \arrow[dr, dashed, "m"] \arrow[drr, "h_2"] \arrow[ddr, "h_1"'] & & \\
& X \arrow[r, "f"'] \arrow[d, "g_f"] & Y \arrow[d, "\chi_f"] \\
& \Omega_1 \arrow[r, "t"'] & \Omega_2
\end{tikzcd}
$$

commutes. Since k is an isomorphism, k^{-1} must be also and therefore $k^{-1} \circ m = n \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(W,\,Z)$. Since $f \circ k = h$ then $f = h \circ k^{-1}$ and therefore $h \circ k^{-1} \circ m = f \circ m = h_2$. Further, since $g_f \circ k = g_h$ then $g_f = g_h \circ k^{-1}$. The last equality gives us the following, $g_h \circ k^{-1} \circ m = g_f \circ m = h_1$. Thus, $g_h \circ n = h_1$ and $h \circ n = h_2$. Suppose we have another morphism $n' : W \to Z$. Then $h_2 = h \circ n' = f \circ k \circ n'$ and $h_1 = g_h \circ n' = g_f \circ k \circ n'$. Thus, $k \circ n' = m$ and $n' = k^{-1} \circ m = n$.

Now we will prove the sufficiency and therefore assume that $\chi_f = \chi_h$ for some g_f and g_h. Then the following diagram can be created.

$$
\begin{tikzcd}
Z \arrow[drr, "h"] \arrow[ddr, "g_h"'] & & \\
& X \arrow[r, "f"'] \arrow[d, "g_f"] & Y \arrow[d, "\chi_f = \chi_h"] \\
& \Omega_1 \arrow[r, "t"'] & \Omega_2
\end{tikzcd}
$$

Since $\chi_f \circ h = \chi_h \circ h = t \circ g_h$ there exists some $k \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Z,\,X)$ such that $f \circ k = h$. The same way we can get a morphism $k' \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X,\,Z)$ such that $h \circ k' = f$. Thus, $(X,\,f) \approx (Z,\,h)$. ∎

4. Factorization of morphisms in the category SET(L)

Suppose we have two objects $X, Y \in \mathtt{Obj}\,\mathrm{SET}(L)$ and some morphism $f \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X,\,Y)$. The map $f : \tilde{X} \to \tilde{Y}$ defines an equivalence relation Q_f on the set $\tilde{X}$, i.e., for all $x,\,y \in \tilde{X}$, $(x,\,y) \in Q_f$ iff $f(x) = f(y)$. Let

$\tilde{Z} = \tilde{X}/Q_f = \{[x] \mid x \in \tilde{X}\}$ where $[x]$ denotes the equivalence class generated by x and let $Z([x]) = \bigvee\{X(u) \mid u \in [x]\}$ (notice that we regard the elements of $\tilde{Z}$ as subsets of the set $\tilde{X}$). Let $\tilde{f} : \tilde{X} \to \tilde{Z}$ be the following, $f(x) = [x]$ for $x \in \tilde{X}$.

Proposition 4.1. $\tilde{f} \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, Z)$.

Proof. Take any $x_0 \in \tilde{X}$. Then

$$Z \circ \tilde{f}(x_0) = Z([x_0]) = \bigvee\{X(u) \mid u \in [x_0]\} \geqslant X(x_0).$$

$\square$

Let $\overline{f} : \tilde{Z} \to \tilde{Y}$ be the following, $\overline{f}([x]) = f(x)$ for $[x] \in \tilde{Z}$.

Proposition 4.2. $\overline{f} \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, Z)$.

Proof. Take any $[x_0] \in \tilde{Z}$. Then $Y \circ \overline{f}([x_0]) = Y \circ f(x_0)$. For every $u \in [x_0]$, $f(u) = f(x_0)$ and therefore $Y \circ f(x_0) = Y \circ f(u) \geqslant X(u)$ since f is a morphism. Thus, $Y \circ f(x_0) \geqslant \bigvee\{X(u) \mid u \in [x_0]\} = Z([x_0])$ and $\overline{f}$ is indeed a morphism. $\square$

One can easily see that the triangle

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\tilde{f}"'] & & Y \\
& Z \arrow[ur, "\overline{f}"'] &
\end{tikzcd}$$

commutes and therefore f factors through Z.

Proposition 4.3. *The morphism* $\overline{f}$ *is an isomorphism iff the following conditions are fulfilled:*

(1) f *is surjective;*
(2) $Y(y) = \bigvee\{X(x) \mid f(x) = y\}$ *for all* $y \in \tilde{Y}$.

Proof. Let us prove the necessity first and therefore assume that $\overline{f}$ is an isomorphism. Clearly, f is surjective. Let us prove that the second condition also holds. Suppose we have some $y_0 \in \tilde{Y}$. Take any $x_0 \in f^{-1}(y_0)$ (notice that $f^{-1}(y_0) \neq \varnothing$ since f is surjective). Then $Y \circ \overline{f}([x_0]) = Z([x_0])$. Further, $Y \circ \overline{f}([x_0]) = Y \circ f(x_0) = Y(y_0)$ and $Z([x_0]) = \bigvee\{X(u) \mid u \in [x_0]\} = \bigvee\{X(u) \mid f(u) = f(x_0) = y_0\}$. Thus, $Y(y_0) = \bigvee\{X(x) \mid f(x) = y_0\}$.

Now we will prove the sufficiency and therefore assume that all conditions of the proposition are fulfilled. Clearly, $\overline{f}$ is bijective. The only thing we have to prove is $Y \circ \overline{f}([x]) = Z([x])$ for all $[x] \in \tilde{Z}$. Suppose we have some $[x_0] \in \tilde{Z}$. Then $Y \circ \overline{f}([x_0]) = Y \circ f(x_0) = \bigvee\{X(x) \mid f(x) = f(x_0)\} = \bigvee\{X(x) \mid x \in [x_0]\} = Z([x_0])$ and therefore $\overline{f}$ is indeed an isomorphism. $\square$

5. Special morphisms

This section is devoted to special morphisms in the category $\mathrm{SET}(L)$. To begin with, we will consider some kinds of monomorphisms.

Suppose we have some morphism $f \in \mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$. Then f is called an extremal monomorphism provided that it satisfies the following two conditions:

(1) f is a monomorphism.
(2) (Extremal condition): If $f = h \circ m$, where m is an epimorphism, then m must be an isomorphism.

The following theorem shows the necessary and sufficient conditions for a morphism to be an extremal monomorphism.

Theorem 5.1. *A morphism $f : X \to Y$ is an extremal monomorphism iff the following conditions are fulfilled:*

(1) *f is injective;*
(2) *$X(x) = Y \circ f(x)$ for all $x \in \tilde{X}$.*

Proof. Let us prove the necessity first and therefore assume that f is an extremal monomorphism. Since f is a monomorphism, f is injective. Further, let $\tilde{Z} = f(\tilde{X})$. For every $z \in \tilde{Z}$ let $Z(z) = Y(z)$. Suppose $m : \tilde{X} \to \tilde{Z}$ is the restriction of the map f to $f(\tilde{X})$. Clearly, $m \in \mathtt{Mor}_{\mathrm{SET}(L)}(X, Z)$ and is surjective. Thus, m is an epimorphism. Suppose $h : \tilde{Z} \to \tilde{Y}$ is the inclusion map, i.e., $h(z) = z$ for every $z \in \tilde{Z}$. Clearly, $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(Z, Y)$ and $f = h \circ m$. Thus, m is an isomorphism that implies, $Z \circ m(x) = X(x)$ for $x \in \tilde{X}$. Take some $x_0 \in \tilde{X}$. Then $Y \circ f(x_0) = Y \circ h \circ m(x_0) = Y \circ m(x_0) = Z \circ m(x_0) = X(x_0)$.

Now we will prove the sufficiency and therefore assume that all conditions of the theorem are fulfilled. Since f is injective, f is a monomorphism. Suppose we have some object $Z \in \mathtt{Obj}\,\mathrm{SET}(L)$ and two morphisms $m : X \to Z$ and $h : Z \to Y$ such that $f = h \circ m$ and m is an epimorphism. Let us prove that m is an isomorphism. Since f is injective m must be also and therefore m is bijective. The only thing we have to verify is $Z \circ m(x) = X(x)$ for every $x \in \tilde{X}$. Suppose we have some $x_0 \in \tilde{X}$. Then $Z \circ m(x_0) \geqslant X(x_0)$ since m is a morphism. Further, since h is a morphism then $X(x_0) = Y \circ f(x_0) = Y \circ h \circ m(x_0) \geqslant Z \circ m(x_0)$. Thus, $Z \circ m(x_0) = X(x_0)$. $\square$

Now we will consider strict monomorphisms in the category $\mathrm{SET}(L)$, i.e., such morphisms $f \in \mathtt{Mor}_{\mathrm{SET}(L)}(X, Y)$ that the following condition is fulfilled: whenever h is a morphism with the property that for all morphisms r and s, $r \circ f = s \circ f$ implies that $r \circ h = s \circ h$ then there exists a unique morphism k

such that the triangle

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ & Z \arrow[ul, dashed, "k"] \arrow[ur, "h"'] & \end{tikzcd}$$

commutes. (Notice that the definition implies, f is a monomorphism.)

THEOREM 5.2. *A morphism $f : X \to Y$ is a strict monomorphism iff the following conditions are fulfilled:*

(1) *f is injective;*
(2) *$X(x) = Y \circ f(x)$ for all $x \in \tilde{X}$.*

PROOF. We will prove the necessity first and therefore assume that f is a strict monomorphism. Then f is a monomorphism and therefore injective. Further, suppose we have some $x_0 \in \tilde{X}$. Since f is a morphism then $X(x_0) \leqslant Y \circ f(x_0)$. Let $\tilde{Z} = \tilde{X}$, $Z(z) = Y \circ f(z)$ and $h = f$. Clearly, $h \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Z, Y)$ and $r \circ h = s \circ h$ whenever $r \circ f = s \circ f$. Thus, there exists a morphism $k \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Z, X)$ such that $f \circ k = h$. Clearly, $k = id_{\tilde{X}}$. Thus, $X \circ k(x_0) = X(x_0) \geqslant Z(x_0) = Y \circ f(x_0)$ and therefore $X(x_0) = Y \circ f(x_0)$.

Now we will prove the sufficiency and therefore assume that all conditions of the theorem are fulfilled. Suppose we have some $h \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Z, Y)$ such that $r \circ h = s \circ h$ whenever $r \circ f = s \circ f$. Let us verify that $h(\tilde{Z}) \subset f(\tilde{X})$. Indeed, suppose there exists some $y_0 \in \tilde{Y}$ such that $y_0 \in h(\tilde{Z})$ and $y_0 \notin f(\tilde{X})$. Let $\tilde{W} = \{w_0, w_1\}$, $W \equiv 1_L$ and $r(\tilde{Y} \backslash \{y_0\}) = s(\tilde{Y}) = \{w_0\}$ but $r(y_0) = w_1$. Clearly, $r \circ f = s \circ f$ but $r \circ h \neq s \circ h$. Now let $k : \tilde{Z} \to \tilde{X}$ be the following, $k(z) = f^{-1} \circ h(z)$ for $z \in \tilde{Z}$. Since f is injective k is defined correctly. Clearly, $f \circ k = f \circ f^{-1} \circ h = h$ and for all other maps k' such that $f \circ k' = h$ it follows that $k = k'$. Let us verify that $k \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Z, X)$. Suppose we have some $z_0 \in \tilde{Z}$. Since h is a morphism then $Y \circ h(z_0) = Y \circ f \circ k(z_0) = X \circ k(z_0) \geqslant Z(z_0)$. Thus, k is indeed a morphism. ∎

Lastly, let us consider one more type of monomorphisms, i.e., strong monomorphisms in the category SET(L). Recall that a morphism $f : X \to Y$ is said to be a strong monomorphism provided that the following two conditions are fulfilled:

(1) f is a monomorphism;

(2) whenever $g \circ m = f \circ k$ with m an epimorphism, there exists a morphism h such that the diagram

$$\begin{tikzcd} Z \arrow[r, "m"] \arrow[d, "k"'] & W \arrow[d, "g"] \arrow[dl, dashed, "h"'] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes.

Theorem 5.3. *A morphism $f : X \to Y$ is a strong monomorphism iff the following conditions are fulfilled:*

(1) *f is injective;*
(2) *$X(x) = Y \circ f(x)$ for all $x \in \tilde{X}$.*

Proof. We will prove the necessity first and therefore assume that f is a strong monomorphism. Then f is a monomorphism and therefore injective. Further, suppose we have some $x_0 \in \tilde{X}$. Then, since f is a morphism, $X(x_0) \leqslant Y \circ f(x_0)$. Let $Z = X$, $\tilde{W} = f(\tilde{X})$, $W(w) = Y(w)$ for $w \in \tilde{W}$ and let m be the restriction of f to $f(\tilde{X})$. Clearly, $m \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Z, W)$ and is surjective, therefore, m is an epimorphism. Let $k = id_{\tilde{X}}$ and let g be the inclusion map, i.e., $g(w) = w$ for $w \in \tilde{W}$. Clearly, both k and g are morphisms and $f \circ k = g \circ m$. Thus, there exists a morphism $h \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(W, X)$ such that $f \circ h = g$. Take $f(x_0) \in \tilde{W}$. Then $f \circ h \circ f(x_0) = g \circ f(x_0) = f(x_0)$ and therefore $h \circ f(x_0) = x_0$ since f is injective. Since h is a morphism then $X \circ h \circ f(x_0) = X(x_0) \geqslant W \circ f(x_0) = Y \circ f(x_0)$. Thus, $X(x_0) = Y \circ f(x_0)$.

Now we will prove the sufficiency and therefore assume that all conditions of the theorem are fulfilled. Since f is injective, f is a monomorphism. Further, suppose we have some morphisms k, m and g such that the square

$$\begin{tikzcd} Z \arrow[r, "m"] \arrow[d, "k"'] & W \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes and m is an epimorphism. Let us verify that $g(\tilde{W}) \subset f(\tilde{X})$. Suppose we have some $y_0 \in g(\tilde{W})$ such that $y_0 \notin f(\tilde{X})$. Then there exists some $w_0 \in \tilde{W}$ such that $g(w_0) = y_0$. Since m is an epimorphism and therefore surjective, there exists some $z_0 \in \tilde{Z}$ such that $m(z_0) = w_0$.

Since $y_0 = g \circ m(z_0) = f \circ k(z_0)$ we have, $f^{-1}(y_0) \neq \varnothing$ that contradicts our former assumption. Now, let $h : \tilde{W} \to \tilde{X}$ be the following, $h(w) = f^{-1} \circ g(w)$ for $w \in \tilde{W}$. Since f is injective h is defined correctly. Clearly, $f \circ h = f \circ f^{-1} \circ g = g$. Further, $f \circ k = g \circ m$ and therefore $k = f^{-1} \circ g \circ m$. Thus, $h \circ m = f^{-1} \circ g \circ m = k$. Let us verify that $h \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(W, X)$. Suppose we have some $w_0 \in \tilde{W}$. Since g is a morphism then $Y \circ g(w_0) = Y \circ f \circ h(w_0) = X \circ h(w_0) \geqslant W(w_0)$. Thus, h is indeed a morphism. □

The last three theorems imply the following result.

THEOREM 5.4. *For every morphism $f \in \mathtt{Mor}\,\mathrm{SET}(L)$ the following are equivalent:*

(1) *f is a regular monomorphism;*
(2) *f is an extremal monomorphism;*
(3) *f is a strict monomorphism;*
(4) *f is a strong monomorphism.*

Now let us consider the dual of monomorphisms, i.e., epimorphisms in the category SET(L). By analogy with monomorphisms we will start by considering extremal epimorphisms.

Suppose we have some morphism $f \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, Y)$. Then f is called an extremal epimorphism provided that it satisfies the following two conditions:

(1) f is a epimorphism.
(2) (Extremal condition): If $f = m \circ h$, where m is a monomorphism, then m must be an isomorphism.

The following theorem shows the necessary and sufficient conditions for a morphism to be an extremal epimorphism.

THEOREM 5.5. *A morphism $f : X \to Y$ is an extremal epimorphism iff the following conditions are fulfilled:*

(1) *f is surjective;*
(2) $Y(y) = \bigvee\{X(x) \,|\, f(x) = y\}$ *for all* $y \in \tilde{Y}$.

PROOF. Let us prove the necessity first and therefore assume that f is an extremal epimorphism. Since f is an epimorphism, f is surjective. Let us take the object Z defined in the previous section. Then f factors through Z. Clearly, $\overline{f}$ is a monomorphism and therefore an isomorphism. The proposition 4.3 implies that all conditions of the theorem are fulfilled.

Now we will prove the sufficiency and therefore assume that all conditions of the theorem are fulfilled. Since f is surjective then f is an epimorphism. Suppose we have some object $Z \in \mathtt{Obj}\,\mathrm{SET}(L)$ and two morphism $h : X \to Z$ and $m : Z \to Y$ such that $m \circ h = f$ and m is a monomorphism. Let us prove that m is an isomorphism. Since f is surjective m must be also and therefore

m is bijective. The only thing we have to verify is $Y \circ m(z) = Z(z)$ for all $z \in \tilde{Z}$. Suppose we have some $z_0 \in \tilde{Z}$. Then $Z(z_0) \leqslant Y \circ m(z_0)$ since m is a morphism. Further, suppose $x_0 \in f^{-1} \circ m(z_0)$. Then $Z \circ h(x_0) \geqslant X(x_0)$ since h is a morphism. Since $f(x_0) = m \circ h(x_0) = m(z_0)$ and m is injective then $h(x_0) = z_0$ and therefore $Z \circ h(x_0) = Z(z_0) \geqslant X(x_0)$. Thus, $Z(z_0) \geqslant \bigvee\{X(x) \mid f(x) = m(z_0)\} = Y \circ m(z_0)$ and therefore $Y \circ m(z_0) = Z(z_0)$. □

Now we will consider strict epimorphisms in the category $\mathrm{SET}(L)$, i.e., such morphisms $f \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, Y)$ that the following condition is fulfilled: whenever h is a morphism with the property that for all morphisms r and s, $f \circ r = f \circ s$ implies that $h \circ r = h \circ s$ then there exists a unique morphism k such that the triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, dashed, "k"] \\ & Z & \end{tikzcd}$$

commutes. (Notice that the definition implies, f is an epimorphism.)

Theorem 5.6. *A morphism $f : X \to Y$ is a strict epimorphism iff the following conditions are fulfilled:*

(1) *f is surjective;*
(2) *$Y(y) = \bigvee\{X(x) \mid f(x) = y\}$ for all $y \in \tilde{Y}$.*

Proof. We will prove the necessity first and therefore assume that f is a strict epimorphism. Then f is an epimorphism and therefore surjective. Further, suppose we have some $y_0 \in \tilde{Y}$. Since f is a morphism, $Y(y_0) \geqslant \bigvee\{X(x) \mid f(x) = y_0\}$. Let $\tilde{Z} = \tilde{Y}$, $Z(z) = \bigvee\{X(x) \mid f(x) = y\}$ and let $h = f$. Clearly, $h \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, Z)$ and $h \circ r = h \circ s$ whenever $f \circ r = f \circ s$. Thus, there exists a morphism $k \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Y, Z)$ such that $k \circ f = h$. Clearly, $k = id_{\tilde{Y}}$. Thus, $Z \circ k(y_0) = Z(y_0) = \bigvee\{X(x) \mid f(x) = y_0\} \geqslant Y(y_0)$ and therefore $Y(y_0) = \bigvee\{X(x) \mid f(x) = y_0\}$.

Now we will prove the sufficiency and therefore assume that all conditions of the theorem are fulfilled. Suppose we have some $h \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, Z)$ such that $h \circ r = h \circ s$ whenever $f \circ r = f \circ s$. Let us verify that $f(x_1) = f(x_2)$ implies $h(x_1) = h(x_2)$ for all $x_1, x_2 \in \tilde{X}$. Indeed, suppose $h(x_1) \neq h(x_2)$ for some $x_1, x_2 \in \tilde{X}$. Then let $\tilde{W} = \{w_0\}$, $W(w_0) = 0_L$ and let $r(w_0) = x_1$, $s(w_0) = x_2$. Since $f \circ r(w_0) = f(x_1) = f(x_2) = f \circ s(w_0)$ then $h(x_1) = h \circ r(w_0) = h \circ s(w_0) = h(x_2)$ that contradicts our former assumption. Now, let $k : \tilde{Y} \to \tilde{Z}$ be the following, $k(y) = h(x)$, $x \in f^{-1}(y)$. Since f is surjective the map k is defined correctly. Clearly, $k \circ f(x) = h(x)$ for all $x \in \tilde{X}$. For all other maps k' such that $k' \circ f = h$ we have, $k' = k$. Let us verify that

$k \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Y, Z)$. Suppose we have some $y_0 \in \tilde{Y}$. Since $f(x_1) = f(x_2)$ implies $h(x_1) = h(x_2)$ then $Z \circ k(y_0) \geqslant \bigvee\{X(x) \mid f(x) = y_0\} = Y(y_0)$. Thus, k is indeed a morphism. ∎

Lastly, let us consider strong epimorphisms in the category SET(L). By analogy with strong monomorphisms a morphism $f \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, Y)$ is said to be a strong epimorphism provided that the following two conditions are fulfilled:

(1) f is an epimorphism;
(2) whenever $g \circ f = m \circ k$ with m a monomorphism, there exists a morphism h such that the diagram

$$\begin{tikzcd}
Z \arrow[r, "m"] & W \\
X \arrow[u, "k"] \arrow[r, "f"'] & Y \arrow[u, "g"'] \arrow[ul, dashed, "h"']
\end{tikzcd}$$

commutes.

Theorem 5.7. *A morphism $f : X \to Y$ is a strong epimorphism iff the following conditions are fulfilled:*

(1) *f is surjective;*
(2) *$Y(y) = \bigvee\{X(x) \mid f(x) = y\}$ for all $y \in \tilde{Y}$.*

Proof. We will prove the necessity first and therefore assume that f is a strong epimorphism. Then f is an epimorphism and therefore surjective. Further, suppose we have some $y_0 \in \tilde{Y}$. Since f is a morphism, $Y(y_0) \geqslant \bigvee\{X(x) \mid f(x) = y_0\}$. Let $W = Y$ and let $g = e_Y$. Further, let us take the object Z defined in the previous section and let $k = \tilde{f}$ and $m = \overline{f}$. Clearly, $g \circ f = m \circ k$ and therefore there exists a morphism $h \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(Y, Z)$ such that $k = h \circ f$. Let us take any $x_0 \in \tilde{X}$ such that $f(x_0) = y_0$. Then $h \circ f(x_0) = h(y_0) = k(x_0) = [x_0]$. Since h is a morphism then $Z \circ h(y_0) = Z([x_0]) = \bigvee\{X(u) \mid u \in [x_0]\} = \bigvee\{X(x) \mid f(x) = y_0\} \geqslant Y(y_0)$ and therefore $Y(y_0) = \bigvee\{X(x) \mid f(x) = y_0\}$.

Now we will prove the sufficiency and therefore assume that all conditions of the theorem are fulfilled. Since f is surjective, f is an epimorphism.

Further, suppose we have some morphism k, m and g such that the square

$$\begin{tikzcd} Z \arrow[r, "m"] & W \\ X \arrow[u, "k"] \arrow[r, "f"'] & Y \arrow[u, "g"'] \end{tikzcd}$$

commutes and m is a monomorphism. Let us verify that $g(\tilde{Y}) \subset m(\tilde{Z})$. Indeed, suppose we have some $w_0 \in \tilde{W}$ such that $w_0 \in g(\tilde{Y})$ and $w_0 \notin m(\tilde{Z})$. Then there exists some $y_0 \in \tilde{Y}$ such that $g(y_0) = w_0$. Since f is surjective there exists some $x_0 \in \tilde{X}$ such that $f(x_0) = y_0$. Thus, $w_0 = g \circ f(x_0) = m \circ k(x_0)$ and therefore $m^{-1}(w_0) \neq \varnothing$ that contradicts our former assumption. Now, let $h : \tilde{Y} \to \tilde{Z}$ be the following, $h(y) = m^{-1} \circ g(y)$ for $y \in \tilde{Y}$. Since m is injective, the map h is defined correctly. Clearly, $m \circ h = m \circ m^{-1} \circ g = g$. Further, $m \circ k = g \circ f$ and therefore $k = m^{-1} \circ g \circ f$. Thus, $h \circ f = m^{-1} \circ g \circ f = k$. Let us verify that $h \in \mathtt{Mor}_{\mathrm{SET}(L)}(Y, Z)$. Suppose we have some $y_0 \in \tilde{Y}$. Since $g \circ f = m \circ k$ and m is a monomorphism then $f(x_1) = f(x_2)$ implies $k(x_1) = k(x_2)$ for all $x_1, x_2 \in \tilde{X}$. Take any $x_0 \in \tilde{X}$ such that $f(x_0) = y_0$. Then $h \circ f(x_0) = h(y_0) = k(x_0)$. Thus, since k is a morphism, $Z \circ k(x_0) = Z \circ h(y_0) \geqslant \bigvee\{X(x) \mid k(x) = h(y_0)\} \geqslant \bigvee\{X(x) \mid f(x) = y_0\} = Y(y_0)$. Thus, h is a morphism. ∎

The last three theorems imply the following result.

Theorem 5.8. *For every morphism $f \in \mathtt{Mor}\,\mathrm{SET}(L)$ the following are equivalent:*

(1) *f is a regular epimorphism;*
(2) *f is an extremal epimorphism;*
(3) *f is a strict epimorphism;*
(4) *f is a strong epimorphism.*

6. Special objects

This section is devoted to some special objects in the category $\mathrm{SET}(L)$. Let us recall some definitions first.

Let $\mathcal{C}$ be a category which has products and let $\mathcal{M}$ be a class of monomorphisms in $\mathcal{C}$. A $\mathcal{C}$-object D is called an $\mathcal{M}$-separator of $\mathcal{C}$ provided that each $\mathcal{C}$-object is an $\mathcal{M}$-subobject of suitable power D^I of D. In particular: (extremal monomorphism)-coseparators are called extremal coseparators and (regular monomorphism)-coseparators are called regular coseparators.

Proposition 6.1. *An object $W \in \mathtt{Obj}\,\mathrm{SET}(L)$, where $\tilde{W} = \{0, 1\}$, $W \equiv 1_L$ is an $\mathcal{M}$-coseparator in $\mathrm{SET}(L)$.*

Proof. Suppose we have some $X \in \mathtt{Obj}\,\mathrm{SET}(L)$. Let us consider the object $W^{\tilde{X}}$. From the properties of product of objects in the category $\mathrm{SET}(L)$ it follows that $\widetilde{W^{\tilde{X}}} = \prod_{x \in \tilde{X}} \tilde{W}_x$, $\tilde{W}_x = \tilde{W}$ and $W^{\tilde{X}}((w_x)_{x \in \tilde{X}}) = \bigwedge_{x \in \tilde{X}} W(w_x)$. Let $f(x_0) = (w_x)_{x \in \tilde{X}}$, where $w_{x_0} = 1$ and $w_x = 0$ for $x \neq x_0$. Clearly, f is injective. Since $W^{\tilde{X}} \circ f(x_0) = 1_L \geqslant X(x_0)$ for all $x_0 \in \tilde{X}$ then $f \in \mathtt{Mor}_{\,\mathrm{SET}(L)}(X, W^{\tilde{X}})$. $\square$

From the theorem 5.4 we derive the following result.

Proposition 6.2. *For an arbitrary object $W \in \mathtt{Obj}\,\mathrm{SET}(L)$ the following are equivalent:*

(1) *W is an extremal coseparator;*
(2) *W is a regular coseparator.*

Now let us consider the dual of coseparators, i.e., separators in the category $\mathrm{SET}(L)$.

Let $\mathcal{C}$ be a category which has coproducts and let $\mathcal{E}$ be a class of epimorphisms in $\mathcal{C}$. A $\mathcal{C}$-object D is called an $\mathcal{E}$-separator of $\mathcal{C}$ provided that each $\mathcal{C}$-object is an $\mathcal{E}$-quotient object of suitable copower ${}^{I}D$ of D. In particular: (extremal epimorphism)-separators are called extremal separators and (regular epimorphism)-separators are called regular separators.

For convenience sake we will consider a subcategory $\mathrm{SET}(L)^*$ of the category $\mathrm{SET}(L)$ where $\tilde{X} \neq \varnothing$ for all $X \in \mathtt{Obj}\,\mathrm{SET}(L)^*$.

Proposition 6.3. *An object $W \in \mathtt{Obj}\,\mathrm{SET}(L)^*$, where $\tilde{W} = \{w_0\}$, $W(w_0) = 0_L$ is an $\mathcal{E}$-separator in the category $\mathrm{SET}(L)^*$.*

Proof. Suppose we have some $X \in \mathtt{Obj}\,\mathrm{SET}(L)^*$. Let us consider the object ${}^{\tilde{X}}W$. From the properties of coproduct of objects in the category $\mathrm{SET}(L)$ it follows that $\widetilde{{}^{\tilde{X}}W} = \bigcup_{x \in \tilde{X}} (w_0, x)$ and ${}^{\tilde{X}}W(w_0, x) = W(w_0) = 0_L$. Let $f : \widetilde{{}^{\tilde{X}}W} \to \tilde{X}$ be the following, $f(w_0, x) = x$ for $x \in \tilde{X}$. Clearly, f is surjective. Since $X \circ f(w_0, x) = X(x) \geqslant 0_L = {}^{\tilde{X}}W(w_0, x)$ for all $x \in \tilde{X}$ then $f \in \mathtt{Mor}_{\,\mathrm{SET}(L)^*}({}^{\tilde{X}}W, X)$. $\square$

Proposition 6.4. *The category $\mathrm{SET}(L)^*$ has no extremal separators.*

Proof. Suppose there exists some $W \in \mathtt{Obj}\,\mathrm{SET}(L)^*$ which is an extremal separator. Let us take two objects $X_1, X_2 \in \mathtt{Obj}\,\mathrm{SET}(L)^*$ where $\tilde{X}_1 = \tilde{X}_2 = \{x_0\}$ and $X_1(x_0) = 0_L$, $X_2(x_0) = 1_L$. Since W is an extremal separator, there exist some sets I_1, I_2 and some extremal epimorphisms $f_1 : {}^{I_1}W \to X_1$, $f_2 : {}^{I_2}W \to X_2$. From the properties of coproduct of objects in the category $\mathrm{SET}(L)$ it follows that $\widetilde{{}^{I_j}W} = \bigcup_{i \in I_j} \tilde{W} \times \{i\}$ and ${}^{I_j}W(w, i) =$

$W(w)$ for all $w \in \tilde{W}$, $j = 1, 2$. Further, since f_1 is an extremal epimorphism then $X_1(x_0) = \bigvee\{^{I_1}W(w, i) \,|\, f_1(w, i) = x_0\} = \bigvee\{W(w) \,|\, f_1(w, i) = x_0\} = \bigvee\limits_{w \in \tilde{W}} W(w)$. The same way we get, $X_2(x_0) = \bigvee\limits_{w \in \tilde{W}} W(w)$. Thus, $0_L = X_1(x_0) = \bigvee\limits_{w \in \tilde{W}} W(w) = X_2(x_0) = 1_L$ that contradicts our definition of lattice. ∎

The theorem 5.8 implies the following result.

PROPOSITION 6.5. *The category* $\mathrm{SET}(L)^*$ *has no regular separators.*

REFERENCES

[1] J.A. Goguen, *L-fuzzy sets*, J. Math. Anal. Appl. **18** (1967), 145-174.

[2] R. Goldblatt, Topoi: The categorical analysis of logic, North-Holland Publ. Co., Amsterdam, 1979.

[3] H. Herrlich and G. Strecker, Category Theory, Heldermann Verlag, Berlin, 1979.

[4] G. Preuß, *Convenient Topology*, Math. Japon **47** (1998), 171-183.

[5] S. Solovyov, *Some remarks on the category SET(L), part I*, to appear in Acta Univ. Latv. (Ser. Math.).

[6] S. Solovyov, *Some remarks on the category SET(L), part II*, to appear in Matematički Vesnik.

[7] L.A. Zadeh, *Fuzzy sets*, Inform. and Control **8** (1965), 338-365.

Department of Mathematics
University of Latvia
Riga LV-1586, Latvia
E-mail: `sergejs@lanet.lv`

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